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Quantum teleportation via a two-qubit Heisenberg XY chain—effects of anisotropy and magnetic field

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Abstract

In this paper we study the influence of anisotropy on the usefulness of the entanglement in a two-qubit Heisenberg XY chain at thermal equilibrium in the presence of an external magnetic field, as a resource for quantum teleportation via the standard teleportation protocol. We show that the nonzero thermal entanglement produced by adjusting the external magnetic field beyond some critical strength is a useful resource. We also consider entanglement teleportation via two two-qubit Heisenberg XY chains.

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1. Introduction

The one-dimensional Heisenberg models have been extensively studied in solid-state physics (see references in [1]). Interest in these models has been revived recently by several proposals for realizing quantum computation [2] and information processing [3] using quantum dots (localized electron spins) as qubits [4]. Lying at the heart of quantum computation and quantum information [5] is a physical resource—quantum entanglement. Consequently, entanglement in interacting Heisenberg spin systems at finite temperatures has been investigated by a number of authors (see, e.g., [6] and references therein).

The state of a typical solid-state system at thermal equilibrium (temperature T) is $\chi = e^{-\beta H}/Z$, where H is the Hamiltonian, $Z = \text{tr}\, e^{-\beta H}$ is the partition function and $\beta = 1/kT$, where k is the Boltzmann constant. The entanglement associated with the thermal state χ is referred to as the thermal entanglement [1]. Due to the availability of a good and computable measure of entanglement for systems of two qubits, the concurrence (see section 2), thermal entanglement in two-qubit Heisenberg spin chains has been thoroughly analysed in terms of thermal concurrence [1, 7–12]. In particular, Kamta *et al* [10] showed that the anisotropy and the magnetic field strength may together be used to control the extent of thermal concurrence in a two-qubit Heisenberg XY chain and, in particular, to produce entanglement for any finite T, by adjusting the external magnetic field beyond some critical strength. The recent breakthroughs in the experimental physics of the double quantum dot (see, e.g., [13]) show that these studies are worthwhile pursuits.

An entangled composite system gives rise to nonlocal correlation between its subsystems that does not exist classically. This nonlocal property enables the use of local quantum operations and classical communication to teleport an unknown quantum state via a shared pair of entangled particles with fidelity (see section 4) better than any classical communication protocol [14–16]. Quantum teleportation can thus serve as an operational test of the presence and 'quality' of entanglement. On the other hand, two-qubit teleportation together with one-qubit unitary operations is sufficient to implement the universal gates for quantum computation [17]. Hence, from both fundamental and practical viewpoints, it is important to study the thermally mixed entangled state of a Heisenberg spin system via quantum teleportation. The possibility of using the thermally mixed entangled state of a two-qubit Heisenberg XX chain as a resource for the standard teleportation protocol $\mathcal{P}_0$ [14] was considered in [18]. The local quantum operations in $\mathcal{P}_0$ consist of Bell measurements and Pauli rotations. It was shown that, although quantum teleportation with fidelity better than any classical communication protocol is possible, the amount of nonzero thermal entanglement does not guarantee this. We could have a more entangled thermal state not achieving a better fidelity than a less entangled one. In fact, the thermally mixed entangled state of a two-qubit Heisenberg XX chain is 'useless' whenever an external magnetic field above some critical strength is applied [18]. Entanglement teleportation [19] using the thermally mixed entangled states of two two-qubit Heisenberg XX chains as resources was also studied in [20].

In view of the above results, we study in this paper the influence of anisotropy on the usefulness, of the entanglement in a two-qubit Heisenberg XY chain at thermal equilibrium in the presence of an external magnetic field, as a resource for quantum teleportation via the standard teleportation protocol $\mathcal{P}_0$. This paper is organized as follows. In section 2, we give the Hamiltonian for the anisotropic two-qubit Heisenberg XY chain, and briefly review a measure of entanglement, the concurrence. We briefly discuss the relevant results of [10] in section 3. This sets the stage necessary for the presentation of our results in section 4. In section 5, we present our results on entanglement teleportation using the thermally mixed entangled states of two two-qubit Heisenberg XY chains as resources. We conclude in section 6.

2. Two-qubit Heisenberg XY chain

The Hamiltonian H for the anisotropic two-qubit Heisenberg XY chain in an external magnetic field $B_m \equiv \eta J$ (η is a real number) along the z-axis is

$$H = \tfrac{1}{2}(1+\gamma)J\sigma_A^1 \otimes \sigma_B^1 + \tfrac{1}{2}(1-\gamma)J\sigma_A^2 \otimes \sigma_B^2 + \tfrac{1}{2}B_m\big(\sigma_A^3 \otimes \sigma_B^0 + \sigma_A^0 \otimes \sigma_B^3\big), \tag{1}$$

where σ_α^0 is the identity matrix and σ_α^i ($i = 1, 2, 3$) are the Pauli matrices at site $\alpha = A, B$. The parameter $-1 \leqslant \gamma \leqslant 1$ measures the anisotropy of the system and equals 0 for the isotropic XX model [7] and ± 1 for the Ising model [9]. $(1+\gamma)J$ and $(1-\gamma)J$ are real coupling constants for the spin interaction. The chain is said to be antiferromagnetic for $J > 0$ and ferromagnetic for $J < 0$.

The eigenvalues and eigenvectors of H are given by [10]

$$\begin{aligned} H|\Phi^0\rangle_{AB} &= \mathcal{B}|\Phi^0\rangle_{AB}, \qquad & H|\Phi^1\rangle_{AB} &= J|\Phi^1\rangle_{AB}, \\ H|\Phi^2\rangle_{AB} &= -J|\Phi^2\rangle_{AB}, \qquad & H|\Phi^3\rangle_{AB} &= -\mathcal{B}|\Phi^3\rangle_{AB}, \end{aligned} \tag{2}$$

where $\mathcal{B} \equiv \sqrt{B_m^2 + \gamma^2 J^2} = \sqrt{\eta^2 + \gamma^2} J$,

$$|\Phi^0\rangle_{AB} = \frac{1}{\sqrt{(\mathcal{B} + B_m)^2 + \gamma^2 J^2}} [(\mathcal{B} + B_m)|00\rangle_{AB} + \gamma J|11\rangle_{AB}], \tag{3}$$

$$|\Phi^1\rangle_{AB} = \frac{1}{\sqrt{2}} [|01\rangle_{AB} + |10\rangle_{AB}], \tag{4}$$

$$|\Phi^2\rangle_{AB} = \frac{1}{\sqrt{2}} [|01\rangle_{AB} - |10\rangle_{AB}], \tag{5}$$

$$|\Phi^3\rangle_{AB} = \frac{1}{\sqrt{(\mathcal{B} - B_m)^2 + \gamma^2 J^2}} [(\mathcal{B} - B_m)|00\rangle_{AB} - \gamma J|11\rangle_{AB}]. \tag{6}$$

When $B_m = 0$, equations (3) and (6) reduce to $\frac{1}{\sqrt{2}}[|00\rangle_{AB} + |11\rangle_{AB}]$ and $\frac{1}{\sqrt{2}}[|00\rangle_{AB} - |11\rangle_{AB}]$ respectively, so that the eigenvectors are the four maximally entangled Bell states: $\left|\Psi^0_{\rm Bell}\right\rangle_{AB}$, $\left|\Psi^1_{\rm Bell}\right\rangle_{AB}$, $\left|\Psi^2_{\rm Bell}\right\rangle_{AB}$ and $\left|\Psi^3_{\rm Bell}\right\rangle_{AB}$. For $\gamma = 0$, $|\Phi^0\rangle_{AB} = |00\rangle_{AB}$ and $|\Phi^3\rangle_{AB} = |11\rangle_{AB}$ with eigenvalues B_m and $-B_m$ respectively, while $|\Phi^1\rangle_{AB}$ and $|\Phi^2\rangle_{AB}$ remain unchanged [7, 18].

To quantify the amount of entanglement associated with a given two-qubit state ρ_{AB}, we consider the concurrence [21, 22] $\mathcal{C}[\rho_{AB}] \equiv \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$, where $\lambda_k (k = 1, 2, 3, 4)$ are the square roots of the eigenvalues in decreasing order of magnitude of the spin-flipped density-matrix operator $R_{AB} = \rho_{AB}\left(\sigma_A^2 \otimes \sigma_B^2\right)\rho^*_{AB}\left(\sigma_A^2 \otimes \sigma_B^2\right)$, the asterisk indicates the complex conjugation. The concurrence associated with the eigenvectors, equations (3) and (6), are given by $\frac{\gamma}{\sqrt{\eta^2+\gamma^2}}$. Hence, they represent entangled states when $\gamma \neq 0$. We note that in the limit of large η,

$$\mathcal{C}[|\Phi^0\rangle_{AB}\langle\Phi^0|] = \mathcal{C}[|\Phi^3\rangle_{AB}\langle\Phi^3|] \approx \gamma\eta^{-1}, \tag{7}$$

going to zero asymptotically when η is infinitely large.

3. Thermal state and concurrence

For the above system in thermal equilibrium at temperature T, its state is described by the density operator

$$\chi_{AB} = \frac{1}{Z}[\mathrm{e}^{-\beta\mathcal{B}}|\Phi^0\rangle_{AB}\langle\Phi^0| + \mathrm{e}^{-\beta J}|\Phi^1\rangle_{AB}\langle\Phi^1| + \mathrm{e}^{\beta J}|\Phi^2\rangle_{AB}\langle\Phi^2| + \mathrm{e}^{\beta\mathcal{B}}|\Phi^3\rangle_{AB}\langle\Phi^3|], \tag{8}$$

where the partition function $Z = 2\cosh\beta\mathcal{B} + 2\cosh\beta J$, the Boltzmann's constant $k \equiv 1$ from hereon, and $\beta = 1/T$. After some straightforward algebra, we obtain

$$\lambda_1 = \frac{1}{Z}\,\mathrm{e}^{\beta J}, \tag{9}$$

$$\lambda_2 = \frac{1}{Z}\,\mathrm{e}^{-\beta J}, \tag{10}$$

$$\lambda_3 = \frac{1}{Z}\sqrt{1 + \frac{2\gamma^2 J^2}{\mathcal{B}^2}\sinh^2\beta\mathcal{B} + \frac{2\gamma J}{\mathcal{B}}\sqrt{1 + \frac{\gamma^2 J^2}{\mathcal{B}^2}\sinh^2\beta\mathcal{B}}\,\sinh\beta\mathcal{B}}, \tag{11}$$

$$\lambda_4 = \frac{1}{Z}\sqrt{1 + \frac{2\gamma^2 J^2}{\mathcal{B}^2}\sinh^2\beta\mathcal{B} - \frac{2\gamma J}{\mathcal{B}}\sqrt{1 + \frac{\gamma^2 J^2}{\mathcal{B}^2}\sinh^2\beta\mathcal{B}}\,\sinh\beta\mathcal{B}}. \tag{12}$$

The concurrence derived from equations (9)–(12) is invariant under the substitutions $\eta \longrightarrow -\eta, \gamma \longrightarrow -\gamma$ and $J \longrightarrow -J$. We shall see that the same applies to the fully entangled fraction (see section 4). Therefore, we restrict our considerations to $\eta \geqslant 0, 0 \leqslant \gamma \leqslant 1$ and $J > 0$.

For $\gamma = 0$, equations (9)–(12) yield the thermal concurrence obtained in [7]:

$$\mathcal{C}[\chi_{AB}] = \max\left\{\frac{\sinh\beta J - 1}{\cosh\beta B_m + \cosh\beta J}, 0\right\}. \tag{13}$$

Equation (13) reduces, when $T = 0$, to

$$\mathcal{C}[\chi_{AB}] = \begin{cases} 1 & \text{for} \quad \eta < 1, \\ \frac{1}{2} & \text{for} \quad \eta = 1, \\ 0 & \text{for} \quad \eta > 1. \end{cases} \tag{14}$$

Hence, with increasing η, the concurrence is constant and maximal, but it drops suddenly to zero as η crosses the critical value $\eta_{\text{critical}} = 1$, marking the point of quantum phase transition (phase transition taking place at zero temperature due to variation of interaction terms in the Hamiltonian of a system [1]). For nonzero T and, $\eta < 1$ or $\eta = 1$, $\mathcal{C}[\chi_{AB}]$ decreases from 1 or $\frac{1}{2}$ respectively to 0 as T increases from 0 to the critical temperature $T^{(1)}_{\text{critical}} \approx 1.134\,59J$. Physically, this is due to mixing of unentangled states and entangled ones. Similarly, when $\eta > 1$, due to mixing of entangled states with the original product state $|11\rangle_{AB}$, $\mathcal{C}[\chi_{AB}]$ increases from 0 to some maximum before decreasing to 0 again at the same $T^{(1)}_{\text{critical}}$. The critical temperature $T^{(1)}_{\text{critical}}$, beyond which $\mathcal{C}[\chi_{AB}] = 0$, is independent of B_m. It also follows from equation (13) that $\mathcal{C}[\chi_{AB}]$ decreases monotonically with increasing B_m for any finite T and vanishes exponentially with increasing B_m, due to an increase in the proportion of $|11\rangle_{AB}$.

For some nonzero γ, equation (8) reduces to the following three possibilities in the zero-temperature limit, i.e., $\beta \longrightarrow \infty$, at which the system is in its ground state.

(a) $0 \leqslant \eta < \sqrt{1-\gamma^2}$:

$$\chi_{AB} = \frac{1}{Z}[\mathrm{e}^{\beta J}|\Phi^2\rangle_{AB}\langle\Phi^2| + \mathrm{e}^{\beta\mathcal{B}}|\Phi^3\rangle_{AB}\langle\Phi^3|] \\ \longrightarrow |\Phi^2\rangle_{AB}\langle\Phi^2|, \tag{15}$$

with $Z = \mathrm{e}^{\beta J} + \mathrm{e}^{\beta\mathcal{B}}$. Equations (9)–(12) give $\mathcal{C}[\chi_{AB}] = 1$, its maximum value, in agreement with the fact that $|\Phi^2\rangle_{AB}$ is a maximally entangled Bell state.

(b) $\eta = \sqrt{1-\gamma^2}$:

$$\chi_{AB} \longrightarrow \tfrac{1}{2}[|\Phi^2\rangle_{AB}\langle\Phi^2| + |\Phi^3\rangle_{AB}\langle\Phi^3|]. \tag{16}$$

From equations (9)–(12), the above equally weighted mixture has

$$\mathcal{C}[\chi_{AB}] = \tfrac{1}{2}(1-\gamma). \tag{17}$$

(c) $\eta > \sqrt{1-\gamma^2}$:

$$\chi_{AB} \longrightarrow |\Phi^3\rangle_{AB}\langle\Phi^3|, \tag{18}$$

and equations (9)–(12) yield accordingly

$$\mathcal{C}[\chi_{AB}] = \frac{\gamma}{\sqrt{\eta^2+\gamma^2}}. \tag{19}$$

Therefore, for a given γ, $\eta_{\text{critical}} = \sqrt{1-\gamma^2}$ marks the point of quantum phase transition. However, in contrast to the isotropic case ($\gamma = 0$) [7], this is not a transition from an entangled phase to an unentangled phase. For values of γ other than $\gamma = \frac{1}{3}$, there is a sudden increase or decrease in $\mathcal{C}[\chi_{AB}]$ at η_{critical}, depending on whether $\gamma > \frac{1}{3}$ or $\gamma < \frac{1}{3}$, before decreasing to zero asymptotically, as η is increased beyond the critical value η_{critical} [10].

For nonzero temperatures, due to mixing, $\mathcal{C}[\chi_{AB}]$ decreases to zero as the temperature T is increased beyond the critical value $T^{(1)}_{\text{critical}}$, as in the isotropic case. However, in contrast, for each nonzero value of γ, $T^{(1)}_{\text{critical}}$ depends on η. It decreases with η when η is increased from 0 to $\eta_{\text{critical}} = \sqrt{1-\gamma^2}$, but increases with η for $\eta > \eta_{\text{critical}}$. In fact, the anisotropy permits one to obtain entangled qubits at higher T and higher B_m than is possible in the isotropic case [10]. An interesting question therefore is whether this entanglement is 'useful' as a resource for teleportation. This is the subject of our next two sections.

4. Teleportation and fully entangled fraction

Standard teleportation $\mathcal{P}_0$ [14] with an arbitrary entangled mixed state resource ρ_{AB} is equivalent to a generalized depolarizing channel $\Lambda_B^{\rho_{AB},\mathcal{P}_0}$, with probabilities given by the maximally entangled components of the resource [23, 24]. Therefore, for the thermally mixed entangled state χ_{AB}, equation (8), we have

$$\Lambda_B^{\chi_{AB},\mathcal{P}_0(m)}(|\psi\rangle_B\langle\psi|) = \sum_{i=0}^{3} {}_{AB}\langle\Psi^i_{\text{Bell}}|\chi_{AB}|\Psi^i_{\text{Bell}}\rangle_{AB} \times \sigma_B^{i\oplus m}|\psi\rangle_B\langle\psi|\sigma_B^{i\oplus m}, \tag{20}$$

where $|\psi\rangle_B = \cos\frac{\vartheta}{2}|0\rangle_B + \mathrm{e}^{i\varphi}\sin\frac{\vartheta}{2}|1\rangle_B$ $(0 \leqslant \vartheta \leqslant \pi, 0 \leqslant \varphi \leqslant 2\pi)$ is an arbitrary unknown pure state of a qubit. Here, $i \oplus m$ denotes summation modulus 4, with $m = 0, 1, 2, 3$. In this paper, reliability for teleportation will be the criterion for judging the quality of the entangled thermal state equation (8). Quantitatively, this is measured by the teleportation fidelity,

$$\Phi\big[\Lambda_B^{\chi_{AB},\mathcal{P}_0(m)}\big] \equiv \int \mathrm{d}\psi\, {}_B\langle\psi|\Lambda_B^{\chi_{AB},\mathcal{P}_0(m)}(|\psi\rangle_B\langle\psi|)|\psi\rangle_B. \tag{21}$$

In the standard teleportation protocol $\mathcal{P}_0$, the maximal teleportation fidelity $\Phi_{\max}\big[\Lambda_B^{\chi_{AB},\mathcal{P}_0}\big]$ achievable is given by [16, 24]

$$\Phi_{\max}\big[\Lambda_B^{\chi_{AB},\mathcal{P}_0}\big] = \frac{2\mathcal{F}[\chi_{AB}]+1}{3}, \tag{22}$$

where the fully entangled fraction

$$\mathcal{F}[\chi_{AB}] \equiv \max_{i=0,1,2,3}\big\{{}_{AB}\langle\Psi^i_{\text{Bell}}|\chi_{AB}|\Psi^i_{\text{Bell}}\rangle_{AB}\big\}. \tag{23}$$

After some straightforward algebra, we obtain

$$\mathcal{F}[\chi_{AB}] = \max\left\{\frac{1}{Z}\mathrm{e}^{\beta J}, \frac{1}{Z}\left(\cosh\beta\mathcal{B} + \frac{\gamma J}{\mathcal{B}}\sinh\beta\mathcal{B}\right), \frac{1}{Z}\mathrm{e}^{-\beta J}, \frac{1}{Z}\left(\cosh\beta\mathcal{B} - \frac{\gamma J}{\mathcal{B}}\sinh\beta\mathcal{B}\right)\right\}. \tag{24}$$

So, $\mathcal{F}[\chi_{AB}]$ is indeed invariant under the substitutions $\eta \longrightarrow -\eta$, $\gamma \longrightarrow -\gamma$ and $J \longrightarrow -J$. Restricting our considerations to $\eta \geqslant 0$, $0 \leqslant \gamma \leqslant 1$ and $J > 0$, we have

$$\mathcal{F}[\chi_{AB}] = \begin{cases} \frac{1}{Z}\mathrm{e}^{\beta J} & \text{for} \quad \sqrt{\eta^2+\gamma^2} \leqslant 1, \\ \frac{1}{Z}\big(\cosh\beta\mathcal{B} + \frac{\gamma J}{\mathcal{B}}\sinh\beta\mathcal{B}\big) & \text{for} \quad \sqrt{\eta^2+\gamma^2} > 1. \end{cases} \tag{25}$$

For a bipartite entangled state ρ_{AB} to be useful for quantum teleportation we must have the fully entangled fraction $\mathcal{F}[\rho_{AB}] > \frac{1}{2}$ [15, 16].

In the zero-temperature limit, equation (25) reduces to

$$\mathcal{F}[\chi_{AB}] = \begin{cases} 1 & \text{for} \quad \sqrt{\eta^2+\gamma^2} < 1, \\ \frac{1}{2} & \text{for} \quad \sqrt{\eta^2+\gamma^2} = 1, \\ \frac{1}{2}\left(1+\frac{\gamma}{\sqrt{\eta^2+\gamma^2}}\right) > \frac{1}{2} & \text{for} \quad \sqrt{\eta^2+\gamma^2} > 1. \end{cases} \tag{26}$$

Therefore, the equally weighted mixture equation (16) is useless for quantum teleportation even though its concurrence equation (17) may not be zero. For a given value of γ, $\eta_{\text{critical}} = \sqrt{1-\gamma^2}$ again marks the point of quantum phase transition from the maximally entangled state equation (15), which yields $\mathcal{F}[\chi_{AB}] = 1$, to the generally nonmaximally entangled state equation (18), which still yields $\mathcal{F}[\chi_{AB}] > \frac{1}{2}$ as long as η is finite. In the limit of large η,

$$\mathcal{F}[\chi_{AB}] \approx \tfrac{1}{2} + \tfrac{1}{2}\gamma\eta^{-1} > \tfrac{1}{2} \tag{27}$$

equals $\frac{1}{2}$ asymptotically when η is infinitely large. This is in contrast to the isotropic case where the quantum phase transition occurs at $\eta_{\text{critical}} = 1$, from the maximally entangled phase to the unentangled phase (see equation (14)). Obviously, in this case $\mathcal{F}[\chi_{AB}]$ jumps from 1 to $\frac{1}{2}$ at $\eta_{\text{critical}} = 1$. And $\mathcal{F}[\chi_{AB}] = \frac{1}{2}$ when η is increased beyond 1 [18].

At nonzero temperatures, due to mixing, $\mathcal{F}[\chi_{AB}]$ decreases to $\frac{1}{2}$ at the critical temperature $T^{(2)}_{\text{critical}}$ beyond which the performance is worse than what classical communication protocol can offer, as in the isotropic case [18]. Also, for each γ, $T^{(2)}_{\text{critical}}$ is dependent on η. To obtain $T^{(2)}_{\text{critical}}$ we consider the following. For the thermal state equation (8) to be useful for quantum teleportation at nonzero T, when $\sqrt{\eta^2+\gamma^2} < 1$, we demand that

$$\sinh\beta J > \cosh\beta\sqrt{\eta^2+\gamma^2}J, \tag{28}$$

and when $\sqrt{\eta^2+\gamma^2} > 1$, we demand that

$$\frac{\gamma}{\sqrt{\eta^2+\gamma^2}}\sinh\beta\sqrt{\eta^2+\gamma^2}J > \cosh\beta J. \tag{29}$$

When $\gamma = 0$, equation (28) can be satisfied as long as $\eta < 1$, but equation (29) is unattainable. $T^{(2)}_{\text{critical}}$ decreases from $1.134\,59J$ to 0 as η is increased from 0 to $\eta_{\text{critical}} = 1$, and remains zero when η is increased beyond 1 [18]. For nonzero γ, $T^{(2)}_{\text{critical}}$ similarly decreases to zero when η is increased from 0 to $\eta_{\text{critical}} = \sqrt{1-\gamma^2}$, as shown in figure 1. However, this is followed by a monotonic increase in $T^{(2)}_{\text{critical}}$ as η is increased beyond $\sqrt{1-\gamma^2}$, in contrast to the isotropic case. The behaviour of $T^{(2)}_{\text{critical}}$ is therefore qualitatively similar to $T^{(1)}_{\text{critical}}$ (compare figure 4 in [10] with our figure 1). In the limit of large η,

$$T^{(2)}_{\text{critical}} \approx \frac{\eta J}{\ln\eta - \ln\gamma + \ln 2}. \tag{30}$$

Therefore, the anisotropy not only allows one to obtain entangled qubits at higher T and higher B_m than is possible in the isotropic case, but also the associated entanglement is useful as a resource for teleportation via $\mathcal{P}_0$.

5. Entanglement teleportation

Lee and Kim [19] considered teleportation of an entangled two-body pure spin-$\frac{1}{2}$ state via two independent, equally entangled, noisy quantum channels represented by Werner states [25]. In their two-qubit teleportation protocol $\mathcal{P}_1$, the joint measurement is decomposable into two independent Bell measurements and the unitary operation into two local one-qubit Pauli

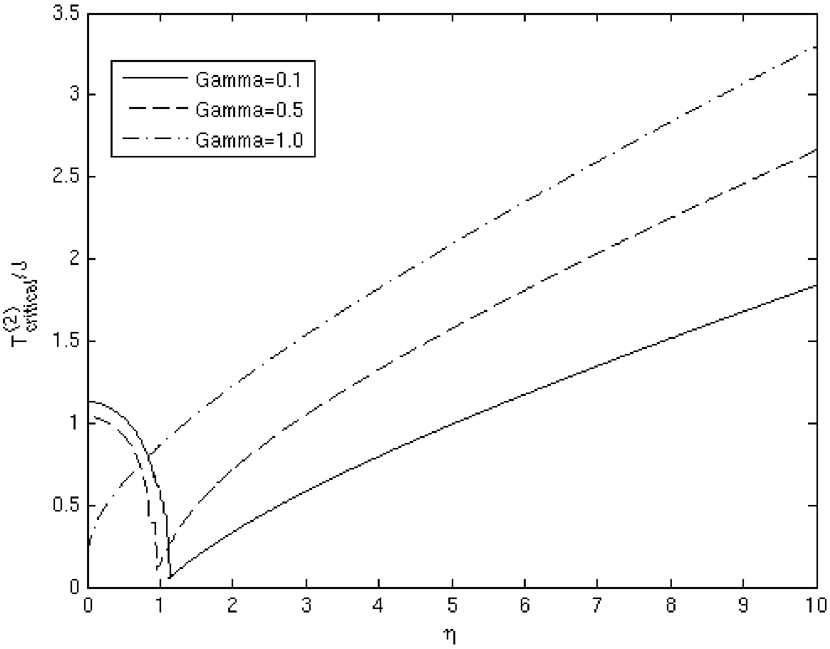

Figure 1. Temperature $T^{(2)}_{\text{critical}}/J$ at which the teleportation fidelity is less than $\frac{2}{3}$, plotted as a function of η (magnetic field strength B_m) for various values of the anisotropy parameter: $\gamma = 0.1$ (solid line), $\gamma = 0.5$ (broken line) and $\gamma = 1$ (dotted line). In each case, the teleportation fidelity is less than or equal to $\frac{2}{3}$ in the region bounded by (and generally above) the relevant curve. Note that for any finite temperature, there is an η for which the fidelity is strictly greater than $\frac{2}{3}$.

rotations. In other words, $\mathcal{P}_1$ is a straightforward generalization of the standard teleportation protocol $\mathcal{P}_0$, just doubling the setup. However, they found that the quantum entanglement of the two-qubit state is lost during the teleportation even when the channel has nonzero quantum entanglement, and in order to teleport quantum entanglement the quantum channels should possess a critical value of minimum entanglement. Hence, teleportating entanglement demands more stringent conditions on the quantum channels. Mathematically, we generalize equation (20) to obtain the teleported (output) state

$$\begin{aligned}\rho^{\text{out}}_{B_1B_2} &\equiv \Lambda^{\chi_{A_1B_1}\otimes\chi_{A_2B_2},\mathcal{P}_1(m,n)}_{B_1B_2}\left(\rho^{\text{in}}_{B_1B_2}\right)\\ &= \sum_{i,j=0}^{3}\left({}_{A_1B_1}\langle\Psi^i_{\text{Bell}}|\chi_{A_1B_1}|\Psi^i_{\text{Bell}}\rangle_{A_1B_1} \times {}_{A_2B_2}\langle\Psi^j_{\text{Bell}}|\chi_{A_2B_2}|\Psi^j_{\text{Bell}}\rangle_{A_2B_2}\right)\\ &\quad\times\left(\sigma^{i\oplus m}_{B_1}\otimes\sigma^{j\oplus n}_{B_2}\right)\rho^{\text{in}}_{B_1B_2}\left(\sigma^{i\oplus m}_{B_1}\otimes\sigma^{j\oplus n}_{B_2}\right),\end{aligned} \tag{31}$$

where $m, n = 0, 1, 2, 3$, and [26]

$$\begin{aligned}\rho^{\text{in}}_{B_1B_2} = \frac{1}{4}\Bigg\{&\sigma^0_{B_1}\otimes\sigma^0_{B_2} + \cos\mu\left(\vec{r}_{B_1}\cdot\vec{\sigma}_{B_1}\otimes\sigma^0_{B_2} + \vec{r}_{B_2}\cdot\sigma^0_{B_1}\otimes\vec{\sigma}_{B_2}\right) + \sum_{i,j=1}^{3}\left[r^i_{B_1}r^j_{B_2}\right.\\ &\left. + \sin\mu\cos\nu\left(k^i_{B_1}k^j_{B_2} - l^i_{B_1}l^j_{B_2}\right) - \sin\mu\sin\nu\left(k^i_{B_1}l^j_{B_2} + l^i_{B_1}k^j_{B_2}\right)\right]\sigma^i_{B_1}\otimes\sigma^j_{B_2}\Bigg\}\end{aligned} \tag{32}$$

is the input state with $\mu, \vartheta_{B_1}, \vartheta_{B_2} \in (0, \pi)$; $\nu, \varphi_{B_1}, \varphi_{B_2} \in (0, 2\pi)$, and

$$\begin{aligned} \vec{r}_\alpha &= (\sin\vartheta_\alpha \cos\varphi_\alpha, \sin\vartheta_\alpha \sin\varphi_\alpha, \cos\vartheta_\alpha), \\ \vec{k}_\alpha &= (\sin\varphi_\alpha, -\cos\varphi_\alpha, 0), \\ \vec{l}_\alpha &= (\cos\vartheta_\alpha \cos\varphi_\alpha, \cos\vartheta_\alpha \sin\varphi_\alpha, -\sin\vartheta_\alpha). \end{aligned} \tag{33}$$

Here $\alpha = B_1$ or B_2. It is easy to verify that $\rho^{in}_{B_1B_2} \cdot \rho^{in}_{B_1B_2} = \rho^{in}_{B_1B_2}$. That is, $\rho^{in}_{B_1B_2}$ is an arbitrary unknown pure state of two qubits, which are in general entangled.

The entanglement teleportation fidelity can be defined analogously to equation (21):

$$\begin{aligned} \Phi\left[\Lambda^{\chi_{A_1B_1}\otimes\chi_{A_2B_2},\mathcal{P}_1(m,n)}_{B_1B_2}\right] \equiv \frac{3}{64\pi^3} \int_0^\pi \int_0^{2\pi} \int_0^\pi \int_0^{2\pi} \int_0^\pi \int_0^{2\pi} \mathrm{d}\mu\,\mathrm{d}\nu\,\mathrm{d}\vartheta_{B_1}\,\mathrm{d}\varphi_{B_1}\,\mathrm{d}\vartheta_{B_2}\,\mathrm{d}\varphi_{B_2} \\ \times \cos^2\mu \sin\mu \sin\vartheta_{B_1} \sin\vartheta_{B_2} \mathrm{tr}\left[\rho^{\mathrm{in}}_{B_1B_2}\rho^{\mathrm{out}}_{B_1B_2}\right]. \end{aligned} \tag{34}$$

After some algebra, we obtain the maximal entanglement teleportation fidelity

$$\Phi_{\max}\left[\Lambda^{\chi_{A_1B_1}\otimes\chi_{A_2B_2},\mathcal{P}_1}_{B_1B_2}\right] = \begin{cases} \frac{1}{5}\left(1 + \frac{4}{Z^2}\mathrm{e}^{2\beta J}\right) & \text{for} \quad \sqrt{\eta^2+\gamma^2} \leqslant 1 \\ \frac{1}{5}\left[1 + \frac{4}{Z^2}\left(\cosh\beta\mathcal{B} + \frac{\gamma J}{\mathcal{B}}\sinh\beta\mathcal{B}\right)^2\right] & \text{for} \quad \sqrt{\eta^2+\gamma^2} > 1. \end{cases} \tag{35}$$

In terms of $\Phi_{\max}\left[\Lambda^{\chi_{A_1B_1}\otimes\chi_{A_2B_2},\mathcal{P}_1}_{B_1B_2}\right]$, for the thermal state equation (8) to be useful for entanglement teleportation, we demand that $\Phi_{\max}\left[\Lambda^{\chi_{A_1B_1}\otimes\chi_{A_2B_2},\mathcal{P}_1}_{B_1B_2}\right] > \frac{2}{5}$ [16]. This condition reduces to equations (28) and (29) respectively. Therefore, no new insights could be gained from the entanglement teleportation fidelity.

It is interesting to analyse the fidelity of the teleported state for some partially unknown pure input state of two qubits, which are in general entangled. Consider, for instance,

$$|\Psi^{in}\rangle_{B_1B_2} = \cos\xi|00\rangle_{B_1B_2} + \sin\xi|11\rangle_{B_1B_2}, \tag{36}$$

where $0 \leqslant \xi \leqslant \pi$, with $\mathcal{C}[|\Psi^{in}\rangle_{B_1B_2}\langle\Psi^{in}|] = |\sin 2\xi|$. For $\sqrt{\eta^2+\gamma^2} \leqslant 1$, we have the maximal output fidelity

$$\begin{aligned} &{}_{B_1B_2}\langle\Psi^{in}|\Lambda^{\chi_{A_1B_1}\otimes\chi_{A_2B_2},\mathcal{P}_1(2,2)}_{B_1B_2}(|\Psi^{in}\rangle_{B_1B_2}\langle\Psi^{in}|)|\Psi^{in}\rangle_{B_1B_2} \\ &\quad = \frac{1}{Z^2}\left[4\cosh^2\beta J + 2\left(1 + \frac{\gamma^2 J^2}{\mathcal{B}^2}\right)\sinh^2\beta\mathcal{B}\sin^2 2\xi\right], \end{aligned} \tag{37}$$

while for $\sqrt{\eta^2+\gamma^2} > 1$,

$$\begin{aligned} &{}_{B_1B_2}\langle\Psi^{in}|\Lambda^{\chi_{A_1B_1}\otimes\chi_{A_2B_2},\mathcal{P}_1(1,1)}_{B_1B_2}(|\Psi^{in}\rangle_{B_1B_2}\langle\Psi^{in}|)|\Psi^{in}\rangle_{B_1B_2} \\ &\quad = \frac{1}{Z^2}\left\{\left(\mathrm{e}^{\beta J} + \cosh\beta\mathcal{B} + \frac{\gamma J}{\mathcal{B}}\sinh\beta\mathcal{B}\right)^2 + \left[\left(\mathrm{e}^{\beta J} + \cosh\beta\mathcal{B} - \frac{\gamma J}{\mathcal{B}}\sinh\beta\mathcal{B}\right)^2 \right.\right. \\ &\qquad \left.\left. - 2\sinh 2\beta J - 4\mathrm{e}^{\beta J}\cosh\beta\mathcal{B}\right]\sin^2 2\xi\right\}. \end{aligned} \tag{38}$$

We note that, in contrast to the result in [19], the relation between the maximal output fidelity and the initial entanglement of the input state is not as straightforward. When $\eta \leqslant \eta_{\mathrm{critical}} = \sqrt{1-\gamma^2}$, the maximal output fidelity always increases monotonically as the initial entanglement increases for any β, γ, η and J. However, if $\eta > \eta_{\mathrm{critical}}$, it could be a monotonic increasing or decreasing function of the initial entanglement, depending on the choice of β, γ, η and J.

6. Conclusions

In conclusion, we show that in an anisotropic two-qubit Heisenberg XY chain, the nonzero thermal entanglement produced by adjusting the external magnetic field beyond some critical strength is a useful resource for teleportation via $\mathcal{P}_0$. It would be interesting to determine if the same occurs in models such as those considered in [11, 12]. We also considered entanglement teleportation via two two-qubit Heisenberg XY chains. In particular, we show that for the partially unknown input state equation (36), the optimal output fidelity could be a monotonic increasing or decreasing function of the entanglement associated with the initial input state.

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